JHEP HYPER VERSION

Received: *December 15, 1998*, Accepted: *January 18, 1998*

Dualities versus singularities

Tom Banks

Department of Physics and Astronomy
Rutgers University, Piscataway, NJ 08855-0849
E-mail: banks@physics.rutgers.edu

Willy Fischler

Theory Group, Department of Physics
University of Texas, Austin, TX 78712
E-mail: fischler@physics.utexas.edu

Luboš Motl

Department of Physics and Astronomy
Rutgers University, Piscataway, NJ 08855-0849
E-mail: motl@physics.rutgers.edu

Abstract: We show that a subgroup of the modular group of M-theory compactified on a ten torus, implies the Lorentzian structure of the moduli space, that is usually associated with naive discussions of quantum cosmology based on the low energy Einstein action. This structure implies a natural division of the asymptotic domains of the moduli space into regions which can/cannot be mapped to Type II string theory or 11D Supergravity (SUGRA) with large radii. We call these the safe and unsafe domains. The safe domain is the interior of the future light cone in the moduli space while the unsafe domain contains the spacelike region and the past light cone. Within the safe domain, apparent cosmological singularities can be resolved by duality transformations and we briefly provide a physical picture of how this occurs. The unsafe domains represent true singularities where all field theoretic description of the physics breaks down. They violate the holographic principle. We argue that this structure provides a natural arrow of time for cosmology. All of the Kasner solutions, of the compactified SUGRA theory interpolate between the past and future light cones of the moduli space. We describe tentative generalizations of this analysis to moduli spaces with less SUSY.

Keywords: M-Theory, Superstring Vacua, String Duality.

JHEP01(1999)019

Contents

1. Introduction

There have been a large number of papers on the application of string theory and M-theory to cosmology [1]-[11]. In the present paper we will study the cosmology of toroidally compactified M-theory, and argue that some of the singularities encountered in the low energy field theory approximation can be resolved by U-duality. We argue that near any of the singularities we study, new light states appear, which cannot be described by the low energy field theory. The matter present in the original universe decays into these states and the universe proceeds to expand into a new large geometry which is described by a different effective field theory.

In the course of our presentation we will have occasion to investigate the moduli space of M-theory with up to ten compactified rectilinear toroidal dimensions and vanishing three form potential. We believe that the results of this investigation are extremely interesting. For tori of dimension, $d \leq 8$, we find that all noncompact regions of the moduli space can be mapped into weakly coupled Type II string theory, or 11D SUGRA, at large volume. For $d \leq 7$ this result is a subset of the results of Witten [12] on toroidal compactification of string theory. The result for $d = 9$ is more interesting. There we find a region of moduli space which cannot be mapped into well understood regions. We argue that a spacetime cannot be in this regime if it satisfies the Bekenstein bound.

For the ten torus the moduli space can be viewed as a 9+1 dimensional Minkowski space. The interior of the future light cone is the region that can be mapped into Type II or 11D (we call the region which can be so mapped the safe domain of the moduli space). We again argue that the other regions violate the Bekenstein bound, in the form of the cosmological holographic principle of [13]. Interestingly, the pure gravity Kasner solutions lie precisely on the light cone in moduli space. The condition that homogeneous perturbations of the Kasner solutions by matter lie inside the light cone of moduli space is precisely that the energy density be positive.

However, every Kasner solution interpolates between the past and future light cones. Thus, M-theory appears to define a natural *arrow of time* for cosmological solutions in the sense that it is reasonable to define the future as that direction in which the universe approaches the safe domain. Cosmological solutions appear to interpolate between a past where the holographic principle cannot be satisfied and a future where it can.

We argue that the $9+1$ dimensional structure, which we derive purely group theoretically,[1] is intimately connected to the De Witt metric on the moduli space. In particular, in the low energy interpretation, the signature of the space is a consequence of the familiar fact that the conformal factor has a negative kinetic energy in the Einstein action. Thus, the fact that the duality group and its moduli space are exact properties of M-theory tells us that this structure of the low energy effective action has a more fundamental significance than one might have imagined. The results of this section are the core of the paper and the reader of limited patience should concentrate his attention on them.

In the section 4 of the paper we speculatively generalize our arguments to moduli spaces with less SUSY. We argue that the proper arena for the study of M-theoretic cosmology is to be found in moduli spaces of compact ten manifolds with three form potential, preserving some SUSY. The duality group is of course the group of discrete isometries of the metric on moduli space. We argue that this is always a $p+1$ dimensional Lorentz manifold, where p is the dimension of the moduli space of appropriate, static, SUSY preserving solutions of 11D SUGRA in 10 compact dimensions, restricted to manifolds of unit volume. We discuss moduli spaces with varying amounts of SUSY, raise questions about the adequacy of the 11D SUGRA picture in less supersymmetric cases, and touch on the vexing puzzle of what it means to speak about a potential on the moduli space in those cases where SUSY allows one. In the appendix we discuss how the even self-dual lattices Γ_8 and $\Gamma_{9,1}$ appear in our framework.

[1] and thus presumably the structure of the (in)famous hyperbolic algebra E_{10}, about which we shall have nothing to say in this paper, besides what is implicit in our use of its Weyl group.

2. Moduli, vacua, quantum cosmology and singularities

2.1 Some idiosyncratic views on general wisdom

M-theorists have traditionally occupied themselves with moduli spaces of Poincaré invariant SUSY vacua. It was hoped that the traditional field theoretic mechanisms for finding stable vacuum states would uniquely pick out a state which resembled the world we observe.

This point of view is very hard to maintain after the String Duality Revolution. It is clear that M-theory has multiparameter families of exact quantum mechanical SUSY ground states, and that the first phenomenological question to be answered by M-theory is why we do not live in one of these SUSY states. It is logical to suppose that the answer to this question lies in cosmology. That is, the universe is as it is not because this is the only stable endpoint to evolution conceivable in M-theory but also because of the details of its early evolution. To motivate this proposal, recall that in infinite Poincaré invariant space time of three or more dimensions, moduli define superselection sectors. Their dynamics is frozen and solving it consists of minimizing some effective potential once and for all, or just setting them at some arbitrary value if there is no potential. Only in cosmology do the moduli become real dynamical variables. Since we are now convinced that Poincaré invariant physics does not destabilize these states, we must turn to cosmology for an explanation of their absence in the world of phenomena.

The focus of our cosmological investigations will be the moduli which are used to parametrize SUSY compactifications of M-theory. We will argue that there is a natural Born-Oppenheimer approximation to the theory in which these moduli are the slow variables. In various semiclassical approximations the moduli arise as zero modes of fields in compactified geometries. One of the central results of String Duality was the realization that various aspects of the space of moduli could be discussed (which aspects depend on precisely how much SUSY there is) even in regions where the notions of geometry, field theory and even weakly coupled string theory, were invalid. *The notion of the moduli space is more robust than its origin in the zero modes of fields would lead us to believe.*

The moduli spaces of solutions of the SUGRA equations of motion that preserve eight or more SUSYs, parametrize exact flat directions of the effective action of M-theory. Thus they can perform arbitrarily slow motions. Furthermore, their action is proportional to the volume of the universe in fundamental units. Thus, once the universe is even an order of magnitude larger than the fundamental scale we should be able to treat the moduli as classical variables. They provide the natural definition of a semiclassical time variable which is necessary to the physical interpretation of a generally covariant theory. In this and the next section we will concentrate on the case with maximal SUSY. We relax this restriction in section 4. There we also discuss briefly the confusing situation of four or fewer SUSYs where there can be a potential on the moduli space.

In this paper we will always use the term moduli in the sense outlined above. They are the slowest modes in the Born-Oppenheimer approximation to M-theory which becomes valid in the regime we conventionally describe by quantum field theory. In this regime they can be thought of as special modes of fields on a large smooth manifold. However, we believe that string duality has provided evidence that the moduli of supersymmetric compactifications are exact concepts in M-theory, while the field theoretic (or perturbative string theoretic) structures from which they were derived are only approximations. The Born-Oppenheimer approximation for the moduli is likely to be valid even in regimes where field theory breaks down. The first task in understanding an M-theoretic cosmology is to discuss the dynamics of the moduli. After that we can begin to ask when and how the conventional picture of quantum field theory in a classical curved spacetime becomes a good approximation.

2.2 Quantum cosmology

The subject of Quantum Cosmology is quite confusing. We will try to be brief in explaining our view of this arcane subject. There are two issues involved in quantizing a theory with general covariance or even just time reparametrization invariance. The first is the construction of a Hilbert space of gauge invariant physical states. The second is describing the physical interpretation of the states and in particular, the notion of time evolution in a system whose canonical Hamiltonian has been set equal to zero as part of the constraints of gauge invariance. The first of these problems has been solved only in simple systems like first quantized string theory or Chern-Simons theory, including pure $2+1$ dimensional Einstein gravity. However, it is a purely mathematical problem, involving no interpretational challenges. We are hopeful that it will be solved in M-theory, at least in principle, but such a resolution clearly must await a complete mathematical formulation of the theory.

The answer to the second problem, depends on a semiclassical approximation. The principle of time reparametrization invariance forces us to base our measurements of time on a physical variable. If all physical variables are quantum mechanical, one cannot expect the notion of time to resemble the one we are used to from quantum field theory. It is well understood [14]-[16] how to derive a conventional time dependent Schrödinger equation for the quantum variables from the semiclassical approximation to the constraint equations for a time reparametrization invariant system. We will review this for the particular system of maximally SUSY moduli.

In fact, we will restrict our attention to the subspace of moduli space described by rectilinear tori with vanishing three form. In the language of 11D SUGRA, we are discussing metrics of the Kasner form

$$ds^2 = -dt^2 + L_i^2(t)\left(dx^i\right)^2, \tag{2.1}$$

where the x^i are ten periodic coordinates with period 1. When restricted to this class of metrics, the Einstein Lagrangian has the form

$$\mathcal{L} = V\left[\sum_i \frac{\dot{L}_i^2}{L_i^2} - \left(\sum_i \frac{\dot{L}_i}{L_i}\right)^2\right], \tag{2.2}$$

where V, the volume, is the product of the L_i. In choosing to write the metric in these coordinates, we have lost the equation of motion obtained by varying the variable g_{00}. This is easily restored by imposing the constraint of time reparametrization invariance. The Hamiltonian E_{00} derived from (2.2) should vanish on physical states. This gives rise to the classical Wheeler-De Witt equation

$$2E_{00} = \left(\sum_i \frac{\dot{L}_i}{L_i}\right)^2 - \sum_i \left(\frac{\dot{L}_i}{L_i}\right)^2 = 0\,, \tag{2.3}$$

which in turn leads to a naive quantum Wheeler-De Witt equation:

$$\frac{1}{4V}\left(\sum_i \Pi_i^2 - \frac{2}{9}(\sum_i \Pi_i)^2\right)\Psi = 0\,. \tag{2.4}$$

That is, we quantize the system by converting the unconstrained phase space variables (we choose the logarithms of the L_i as canonical coordinates) to operators in a function space. Then physical states are functions satisfying the partial differential equation (2.4). There are complicated mathematical questions involved in constructing an appropriate inner product on the space of solutions, and related problems of operator ordering. In more complex systems it is essential to use the BRST formalism to solve these problems. We are unlikely to be able to resolve these questions before discovering the full nonperturbative formulation of M-theory. However, for our present semiclassical considerations these mathematical details are not crucial.

We have already emphasized that when the volume of the system is large compared to the Planck scale, the moduli behave classically. It is then possible to use the time defined by a particular classical solution (in a particular coordinate system in which the solution is nonsingular for almost all time). Mathematically what this means is that in the large volume limit, the solution to the Wheeler De Witt equation takes the form

$$\psi_{WKB}(c)\Psi(q, t[c_0])\,. \tag{2.5}$$

Here c is shorthand for the variables which are treated by classical mechanics, q denotes the rest of the variables and c_0 is some function of the classical variables which is a monotonic function of time. The wave function Ψ satisfies a time dependent Schrödinger equation

$$i\partial_t\Psi = H(t)\Psi\,, \tag{2.6}$$

and it is easy to find an inner product which makes the space of its solutions into a Hilbert space and the operators $H(t)$ Hermitian. In the case where the quantum variables q are quantum fields on the geometry defined by the classical solution, this approximation is generally called Quantum Field Theory in Curved Spacetime. We emphasize however that the procedure is very general and depends only on the validity of the WKB approximation for at least one classical variable, and the fact that the Wheeler De Witt equation is a second order hyperbolic PDE, with one timelike coordinate. These facts are derived in the low energy approximation of M-theory by SUGRA. However, we will present evidence in the next section that they are consequences of the U-duality symmetry of the theory and therefore possess a validity beyond that of the SUGRA approximation.

From the low energy point of view, the hyperbolic nature of the equation is a consequence of the famous negative sign for the kinetic energy of the conformal factor in Einstein gravity, and the fact that the kinetic energies of all the other variables are positive. It means that the moduli space has a Lorentzian metric.

2.3 Kasner singularities and U-duality

The classical Wheeler-De Witt-Einstein equation for Kasner metrics takes the form:

$$\left(\frac{\dot{V}}{V}\right)^2 - \sum_{i=1}^{10}\left(\frac{\dot{L}_i}{L_i}\right)^2 = 0\,. \tag{2.7}$$

This should be supplemented by equations for the ratios of individual radii R_i; $\prod_{i=1}^{10} R_i = 1$. The latter take the form of geodesic motion with friction on the manifold of R_i (which we parametrize *e.g.* by the first nine ratios)

$$\partial_t^2 R_i + \Gamma^i_{jk}\partial_t R_j \partial_t R_k + \partial_t(\mathrm{ln}V)\partial_t R_i\,, \tag{2.8}$$

Γ is the Christoffel symbol of the metric G_{ij} on the unimodular moduli space. We write the equation in this general form because many of our results remain valid when the rest of the variables are restored to the moduli space, and even generalize to the less supersymmetric moduli spaces discussed in section 4. By introducing a new time variable through $V(t)\partial_t = -\partial_s$ we convert this equation into nondissipative geodesic motion on moduli space. Since the "energy" conjugate to the variable s is conserved, the energy of the nonlinear model in cosmic time (the negative term in the Wheeler De Witt equation) satisfies

$$\partial_t E = -2\partial_t(\ln V)E\,, \tag{2.9}$$

whose solution is

$$E = \frac{E_0}{V^2}\,. \tag{2.10}$$

Plugging this into the Wheeler De Witt equation we find that $V \sim t$ (for solutions which expand as $t \to \infty$). Thus, for this class of solutions we can choose the volume as the monotonic variable c_0 which defines the time in the quantum theory.

For the Kasner moduli space, we find that the solution of the equations for individual radii are

$$R_i(t) = L_{planck}(t/t_0)^{p_i}\,, \tag{2.11}$$

where

$$\sum p_i^2 = \sum p_i = 1\,. \tag{2.12}$$

Note that the equation (2.12) implies that at least one of the p_i is negative (we have again restricted attention to the case where the volume expands as time goes to infinity).

It is well known that all of these solutions are singular at both infinite and zero time. Note that if we add a matter or radiation energy density to the system then it dominates the system in the infinite volume limit and changes the solutions for the geometry there. However, near the singularity at vanishing volume both matter and radiation become negligible (despite the fact that their densities are becoming infinite) and the solutions retain their Kasner form.

All of this is true in 11D SUGRA. In M-theory we know that many regions of moduli space which are apparently singular in 11D SUGRA can be reinterpreted as living in large spaces described by weakly coupled Type II string theory or a dual version of 11D SUGRA. The vacuum Einstein equations are of course invariant under these U-duality transformations. So one is lead to believe that many apparent singularities of the Kasner universes are perfectly innocuous.

Note however that phenomenological matter and radiation densities which one might add to the equations are not invariant under duality. The energy density truly becomes singular as the volume goes to zero. How then are we to understand the meaning of the duality symmetry? The resolution is as follows. We know that when radii go to zero, the effective field theory description of the universe in 11D SUGRA becomes singular due to the appearance of new low frequency states. We also know that the singularity in the energy densities of matter and radiation implies that scattering cross sections are becoming large. Thus, it seems inevitable that phase space considerations will favor the rapid annihilation of the existing energy densities into the new light degrees of freedom. This would be enhanced for Kaluza-Klein like modes, whose individual energies are becoming large near the singularity.

Thus, near a singularity with a dual interpretation, the contents of the universe will be rapidly converted into new light modes, which have a completely different view of what the geometry of space is.[2] The most effective description of the new situation is in terms of the transformed moduli and the new light degrees of freedom. The latter can be described in terms of fields in the reinterpreted geometry. We want to emphasize strongly the fact that the moduli do not change in this transformation, but are merely reinterpreted. This squares with our notion that they are exact

[2] After this work was substantially complete, we received a paper, [17], which proposes a similar view of certain singularities. See also [18].

concepts in M-theory. By contrast, the fields whose zero modes they appear to be in a particular semiclassical regime, do not always make sense. The momentum modes of one interpretation are brane winding modes in another and there is no approximate way in which we can consider both sets of local fields at the same time. Fortunately, there is also no regime in which both kinds of modes are at low energy simultaneously, so in every regime where the time dependence is slow enough to make a low energy approximation, we can use local field theory.

This mechanism for resolving cosmological singularities leads naturally to the question of precisely which noncompact regions of moduli space can be mapped into what we will call the *safe domain* in which the theory can be interpreted as either 11D SUGRA or Type II string theory with radii large in the appropriate units. The answer to this question is, we believe, more interesting than the idea which motivated it. We now turn to the study of the moduli space.

3. The moduli space of M-theory on rectangular tori

In this section, we will study the structure of the moduli space of M-theory compactified on various tori T^k with $k \leq 10$. We are especially interested in noncompact regions of this space which might represent either singularities or large universes. As above, the three-form potential A_{MNP} will be set to zero and the circumferences of the cycles of the torus will be expressed as the exponentials

$$\frac{L_i}{L_{planck}} = t^{p_i}\,, \qquad i = 1, 2, \ldots, k\,. \tag{3.1}$$

The remaining coordinates x^0 (time) and $x^{k+1} \ldots x^{10}$ are considered to be infinite and we never dualize them.

So the radii are encoded in the logarithms p_i. We will study limits of the moduli space in various directions which correspond to keeping p_i fixed and sending $t \to \infty$ (the change to $t \to 0$ is equivalent to $p_i \to -p_i$ so we do not need to study it separately).

We want to emphasize that our discussion of asymptotic domains of moduli space is complete, even though we restrict ourselves to rectilinear tori with vanishing three form. Intuitively this is because the moduli we leave out are angle variables. More formally, the full moduli space is a homogeneous space. Asymptotic domains of the space correspond to asymptotic group actions, and these can always be chosen in the Cartan subalgebra. The p_i above can be thought of as parametrizing a particular Cartan direction in E_{10}.[3]

[3]We thank E. Witten for a discussion of this point.

3.1 The 2/5 transformation

M-theory has dualities which allows us to identify the vacua with different p_i's. A subgroup of this duality group is the S_k which permutes the p_i's. Without loss of generality, we can assume that $p_1 \leq p_2 \leq \ldots \leq p_9$. We will assume this in most of the text. The full group that leaves invariant rectilinear tori with vanishing three form is the Weyl group of the noncompact E_k group of SUGRA. We will denote it by $\mathbb{G}_k$. We will give an elementary derivation of the properties of this group for the convenience of the reader. Much of this is review, but our results about the boundaries of the fundamental domain of the action of $\mathbb{G}_k$ with $k = 9, 10$ on the moduli space, are new. $\mathbb{G}_k$ is generated by the permutations, and one other transformation which acts as follows:

$$(p_1, p_2, \ldots, p_k) \mapsto \left(p_1 - \frac{2s}{3}, p_2 - \frac{2s}{3}, p_3 - \frac{2s}{3}, p_4 + \frac{s}{3}, \ldots, p_k + \frac{s}{3}\right). \tag{3.2}$$

where $s = (p_1 + p_2 + p_3)$. Before explaining why this transformation is a symmetry of M-theory, let us point out several of its properties (3.2).

1. The total sum $S = \sum_{i=1}^{k} p_i$ changes to $S \mapsto S + (k-9)s/3$. So if $s < 0$, the sum increases for $k < 9$, decreases for $k > 9$ and is left invariant for $k = 9$.

2. If we consider all p_i's to be integers which are equal modulo 3, this property will hold also after the 2/5 transformation. The reason is that, due to the assumptions, s is a multiple of three and the coefficients $-2/3$ and $+1/3$ differ by an integer. As a result, from any initial integer p_i's we get p_i's which are multiples of 1/3 which means that all the matrix elements of matrices in $\mathbb{G}_k$ are integer multiples of 1/3.

3. The order of p_1, p_2, p_3 is not changed (the difference $p_1 - p_2$ remains constant, for instance). Similarly, the order of $p_4, p_5, \ldots, p_k$ is unchanged. However the ordering between $p_{1\ldots3}$ and $p_{4\ldots k}$ can change in general. By convention, we will follow each 2/5 transformation by a permutation which places the p_i's in ascending order.

4. The bilinear quantity $I = (9-k)\sum(p_i^2) + (\sum p_i)^2 = (10-k)\sum(p_i^2) + 2\sum_{i<j} p_i p_j$ is left invariant by $\mathbb{G}_k$.

The fact that 2/5 transformation is a symmetry of M-theory can be proved as follows. Let us interpret L_1 as the M-theoretical circle of a type IIA string theory. Then the simplest duality which gives us a theory of the same kind (IIA) is the double T-duality. Let us perform it on the circles L_2 and L_3. The claim is that if we combine this double T-duality with a permutation of L_2 and L_3 and interpret the new L_1 as the M-theoretical circle again, we get precisely (3.2).

Another illuminating way to view the transformation 2/5 transformation is to compactify M-theory on a three torus. The original M2-brane and the M5-brane wrapped on the three torus are both BPS membranes in eight dimensions. One can argue that there is a duality transformation exchanging them [19]. In the limit in which one of the cycles of the T^3 is small, so that a type II string description becomes appropriate, it is just the double T-duality of the previous paragraph. The fact that this transformation plus permutations generates $\mathbb{G}_k$ was proven by the authors of [20] for $k \leq 9$, see also [21].

3.2 Extreme moduli

There are three types of boundaries of the toroidal moduli space which are amenable to detailed analysis. The first is the limit in which eleven-dimensional supergravity becomes valid. We will denote this limit as 11D. The other two limits are weakly coupled type IIA and type IIB theories in 10 dimensions. We will call the domain of asymptotic moduli space which can be mapped into one of these limits, the safe domain.

1. For the limit 11D, all the radii must be greater than L_{planck}. Note that for $t \to \infty$ it means that all the radii are much greater than L_{planck}. In terms of the p_i's , this is the inequality $p_i > 0$.

2. For type IIA, the dimensionless coupling constant g_s^{IIA} must be smaller than 1 (much smaller for $t \to \infty$) and all the remaining radii must be greater than L_{string} (much greater for $t \to \infty$).

3. For type IIB, the dimensionless coupling constant g_s^{IIB} must be smaller than 1 (much smaller for $t \to \infty$) and all the remaining radii must be greater than L_{string} (much greater for $t \to \infty$), including the extra radius whose momentum arises as the number of wrapped M2-branes on the small T^2 in the dual 11D SUGRA picture.

If we assume the canonical ordering of the radii, i.e. $p_1 \leq p_2 \leq p_3 \leq \ldots \leq p_k$, we can simplify these requirements as follows:

1. 11D: $0 < p_1$,

2. IIA: $p_1 < 0 < p_1 + 2p_2$,

3. IIB: $p_1 + 2p_2 < 0 < p_1 + 2p_3$.

To derive this, we have used the familiar relations:

$$\frac{L_1}{L_{planck}} = \left(g_s^{IIA}\right)^{2/3} = \left(\frac{L_{planck}}{L_{string}}\right)^2 = \left(\frac{L_1}{L_{string}}\right)^{2/3} \tag{3.3}$$

for the 11D/IIA duality (L_1 is the M-theoretical circle) and similar relations for the 11D/IIB case ($L_1 < L_2$ are the parameters of the T^2 and L_{IIB} is the circumference of the extra circle):

$$\frac{L_1}{L_2} = g_s^{IIB}, \qquad 1 = \frac{L_1 L_{string}^2}{L_{planck}^3} = \frac{g_s^{IIB} L_2 L_{string}^2}{L_{planck}^3} = \frac{L_{IIB} L_1 L_2}{L_{planck}^3}, \tag{3.4}$$

$$\frac{1}{g_s^{IIB}} \left(\frac{L_{planck}}{L_{string}}\right)^4 = \frac{L_1 L_2}{L_{planck}^2} = \frac{L_{planck}}{L_{IIB}} = \left(g_s^{IIB}\right)^{1/3} \left(\frac{L_{string}}{L_{IIB}}\right)^{4/3} \tag{3.5}$$

Note that the regions defined by the inequalities above cannot overlap, since the regions are defined by $M, M^c \cap A, A^c \cap B$ where A^c means the complement of a set. Furthermore, assuming $p_i < p_{i+1}$ it is easy to show that $p_1 + 2p_3 < 0$ implies $p_1 + 2p_2 < 0$ and $p_1 + 2p_2 < 0$ implies $3p_1 < 0$ or $p_1 < 0$.

This means that (neglecting the boundaries where the inequalities are saturated) the region outside $11\mathrm{D} \cup \mathrm{IIA} \cup \mathrm{IIB}$ is defined simply by $p_1 + 2p_3 < 0$. The latter characterization of the safe domain of moduli space will simplify our discussion considerably.

The invariance of the bilinear form defined above gives an important constraint on the action of $\mathbb{G}_k$ on the moduli space. For $k = 10$ it is easy to see that, considering the p_i to be the coordinates of a ten vector, it defines a Lorentzian metric on this ten dimensional space. Thus the group $\mathbb{G}_{10}$ is a discrete subgroup of $O(1,9)$. The direction in this space corresponding to the sum of the p_i is timelike, while the hyperplane on which this sum vanishes is spacelike. We can obtain the group $\mathbb{G}_9$ from the group $\mathbb{G}_{10}$ by taking p_{10} to infinity and considering only transformations which leave it invariant. Obviously then, $\mathbb{G}_9$ is a discrete subgroup of the transverse Galilean group of the infinite momentum frame. For $k \leq 8$ on the other hand, the bilinear form is positive definite and $\mathbb{G}_k$ is contained in $O(k)$. Since the latter group is compact, and there is a basis in which the $\mathbb{G}_k$ matrices are all integers divided by 3, we conclude that in these cases $\mathbb{G}_k$ is a finite group. In a moment we will show that $\mathbb{G}_9$ and *a fortiori* $\mathbb{G}_{10}$ are infinite. Finally we note that the 2/5 transformation is a spatial reflection in $O(1,9)$. Indeed it squares to 1 so its determinant is ± 1. On the other hand, if we take all but three coordinates very large, then the 2/5 transformation of those

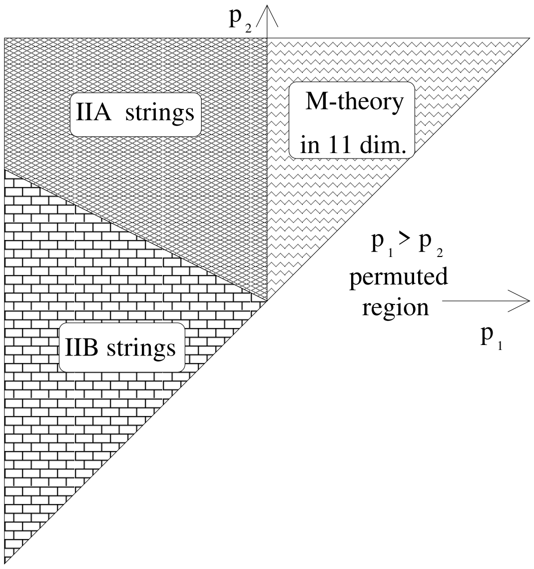

Figure 1: The structure of the moduli space for T^2.

coordinates is very close to the spatial reflection through the plane $p_1+p_2+p_3=0$, so it is a reflection of a single spatial coordinate.

We now prove that $\mathbb{G}_9$ is infinite. Start with the first vector of p_i's given below and iterate (3.2) on the three smallest radii (a strategy which we will use all the time) – and sort p_i's after each step, so that their index reflects their order on the real line. We get

$$\begin{array}{l}(-1,-1,-1,\quad -1,-1,-1,\quad -1,-1,-1)\\(-2,-2,-2,\quad -2,-2,-2,\quad +1,+1,+1)\\(-4,-4,-4,\quad -1,-1,-1,\quad +2,+2,+2)\\(-5,-5,-5,\quad -2,-2,-2,\quad +4,+4,+4)\\ \qquad\qquad\qquad\vdots\\(3\times(2-3n),\ \ 3\times(-1),\ \ 3\times(3n-4))\\(3\times(1-3n),\ \ 3\times(-2),\ \ 3\times(3n-2))\end{array} \tag{3.6}$$

so the entries grow (linearly) to infinity.

3.3 Covering the moduli space

We will show that there is a useful strategy which can be used to transform any point $\{p_i\}$ into the safe domain in the case of T^k, $k<9$. The strategy is to perform iteratively 2/5 transformations on the three smallest radii.

Assuming that $\{p_i\}$ is outside the safe domain, i.e. $p_1+2p_3<0$ (p_i's are sorted so that $p_i\le p_{i+1}$), it is easy to see that $p_1+p_2+p_3<0$ (because $p_2\le p_3$). As we said below the equation (3.2), the 2/5 transformation on p_1,p_2,p_3 always increases the total sum $\sum p_i$ for $p_1+p_2+p_3<0$. But this sum cannot increase indefinitely because the group $\mathbb{G}_k$ is finite for $k<9$. Therefore the iteration proccess must terminate at some point. The only way this can happen is that the assumption $p_1+2p_3<0$ no longer holds, which means that we are in the safe domain. This completes the proof for $k<9$.

For $k=9$ the proof is more difficult. The group $\mathbb{G}_9$ is infinite and furthermore, the sum of all p_i's does not change. In fact the conservation of $\sum p_i$ is the reason that only points with $\sum p_i>0$ can be dualized to the safe domain. The reason is that if $p_1+2p_3\ge 0$, also $3p_1+6p_3\ge 0$ and consequently

$$p_1+p_2+p_3+p_4+p_5+p_6+p_7+p_8+p_9\ge p_1+p_1+p_1+p_3+p_3+p_3+p_3+p_3+p_3\ge 0. \tag{3.7}$$

This inequality is saturated only if all p_i's are equal to each other. If their sum vanishes, each p_i must then vanish. But we cannot obtain a zero vector from a nonzero vector by 2/5 transformations because they are nonsingular. If the sum $\sum p_i$ is negative, it is also clear that we cannot reach the safe domain.

However, if $\sum_{i=1}^{9}p_i>0$, then we can map the region of moduli space with $t\to\infty$ to the safe domain. We will prove it for rational p_i's only. This assumption compensates for the fact that the order of $\mathbb{G}_9$ is infinite. Assuming p_i's rational is however sufficient because we will see that a finite product of 2/5 transformations brings us to the safe domain. But a composition of a finite number of 2/5 transformations is a continuous map from $\mathbb{R}^9$ to $\mathbb{R}^9$ so there must be at least a "ray" part of a neighborhood which can be also dualized to the safe domain. Because $\mathbb{Q}^9$ is dense in $\mathbb{R}^9$, our argument proves the result for general values of p_i.

From now on we assume that the p_i's are rational numbers. Everything is scale invariant so we may multiply them by a common denominator to make integers. In fact, we choose them to be integer multiples of three since in that case we will have integer p_i's even after 2/5 transformations. The numbers p_i are now integers equal modulo 3 and their sum is positive. We will define a critical quantity

$$C = \sum_{i<j}^{1...9} (p_i - p_j)^2 \,. \tag{3.8}$$

This is *a priori* an integer greater than or equal to zero which is invariant under permutations. What happens to C if we make a 2/5 transformation on the radii p_1, p_2, p_3? The differences $p_1 - p_2$, $p_1 - p_3$, $p_2 - p_3$ do not change and this holds for $p_4 - p_5, \ldots p_8 - p_9$, too. The only contributions to (3.8) which are changed are from $3 \cdot 6 = 18$ "mixed" terms like $(p_1 - p_4)^2$. Using (3.2),

$$(p_1 - p_4) \mapsto \left(p_1 - \frac{2s}{3}\right) - \left(p_4 + \frac{s}{3}\right) = (p_1 - p_4) - s\,, \tag{3.9}$$

so its square

$$(p_1 - p_4)^2 \mapsto [(p_1 - p_4) - s]^2 = (p_1 - p_4)^2 - 2s(p_1 - p_4) + s^2 \tag{3.10}$$

changes by $-2s(p_1 - p_4) + s^2$. Summing over all 18 terms we get $(s = p_1 + p_2 + p_3)$

$$\begin{aligned}\Delta C &= -2s\Big[6(p_1 + p_2 + p_3) - 3(p_4 + \ldots + p_9)\Big] + 18s^2 \\ &= 6s^2 + 6\left(\left(\sum_{i=1}^{9} p_i\right) - s\right) \\ &= 6s \sum_{i=1}^{9} p_i \,. \end{aligned} \tag{3.11}$$

But this quantity is strictly negative because $\sum p_i$ is positive and $s < 0$ (we define the safe domain with boundaries, $p_1 + 2p_3 \geq 0$).

This means that C defined in (3.8) decreases after each 2/5 transformation on the three smallest radii. Since it is a non-negative integer, it cannot decrease indefinitely. Thus the assumption $p_1 + 2p_3 < 0$ becomes invalid after a finite number of steps and we reach the safe domain.

The mathematical distinction between the two regions of the moduli space according to the sign of the sum of nine p_i's, has a satisfying interpretation in terms of the holographic principle. In the safe domain, the volume of space grows in the appropriate Planck units, while in the region with negative sum it shrinks to zero. The holographic principle tells us that in the former region we are allowed to describe many of the states of M-theory in terms of effective field theory while in the latter region we are not. The two can therefore not be dual to each other.

Now let us turn to the fully compactified case. As we pointed out, the bilinear form $I \equiv 2\sum_{i<j} p_i p_j$ defines a Lorentzian signature metric on the vector space whose components are the p_i. The $2/5$ transformation is a spatial reflection and therefore the group $\mathbb{G}_{10}$ consists of orthochronous Lorentz transformations. Now consider a vector in the safe domain. We can write it as

$$(-2, -2+a_1, 1+a_2, \ldots, 1+a_9)S\,, \qquad S \in \mathbb{R}^+\,, \tag{3.12}$$

where the a_i are positive. It is easy to see that I is positive on this configuration. This means that only the inside of the light cone can be mapped into the safe domain. Furthermore, since $\sum p_i$ is positive in the safe domain and the transformations are orthochronous, only the interior of the future light cone in moduli space can be mapped into the safe domain.

We would now like to show that the entire interior of the forward light cone can be so mapped. We use the same strategy of rational coordinates dense in $\mathbb{R}^{10}$. If we start outside the safe domain, the sum of the first three p_i is negative. We again pursue the strategy of doing a $2/5$ transformation on the first three coordinates and then reordering and iterating. For the case of $\mathbb{G}_9$ the sum of the coordinates was an invariant, but here it decreases under the $2/5$ transformation of the three smallest coordinates, if their sum is negative. But $\sum p_i$ is (starting from rational values and rescaling to get integers congruent modulo three as before) a positive integer and must remain so after $\mathbb{G}_{10}$ operations. Thus, after a finite number of iterations, the assumption that the sum of the three smallest coordinates is negative must fail, and we are in the safe domain. In fact, we generically enter the safe domain before this point. The complement of the safe domain always has negative sum of the first three coordinates, but there are elements in the safe domain where this sum is negative.

It is quite remarkable that the bilinear form I is proportional to the Wheeler-De Witt Hamiltonian for the Kasner solutions:

$$\frac{I}{t^2} = \left(\sum_i \frac{dL_i/dt}{L_i}\right)^2 - \sum_i \left(\frac{dL_i/dt}{L_i}\right)^2 = \frac{2}{t^2}\sum_{i<j} p_i p_j\,. \tag{3.13}$$

The solutions themselves thus lie precisely on the future light cone in moduli space. Each solution has two asymptotic regions ($t \to 0, \infty$ in (2.1)), one of which is in the past light cone and the other in the future light cone of moduli space. The structure of the modular group thus suggests a natural arrow of time for cosmological evolution. The future may be defined as the direction in which the solution approaches the safe domain of moduli space. All of the Kasner solutions then, have a true singularity in their past, which cannot be removed by duality transformations.

Actually, since the Kasner solutions are on the light cone, which is the boundary of the safe domain, we must add a small homogeneous energy density to the system in order to make this statement correct. The condition that we can map into the safe domain is then the statement that this additional energy density is positive. Note that in the safe domain, and if the equation of state of this matter satisfies (but does not saturate) the holographic bound of [13], this energy density dominates the evolution of the universe, while near the singularity, it becomes negligible compared to the Kasner degrees of freedom. The assumption of a homogeneous negative energy density is manifestly incompatible with Einstein's equations in a compact flat universe so we see that the spacelike domain of moduli space corresponds to a physical situation which cannot occur in the safe domain.

The backward lightcone of the asymptotic moduli space is, as we have said, visited by all of the classical solutions of the theory. However, it violates the holographic principle of [13] if we imagine that the universe has a constant entropy density per comoving volume. We emphasize that in this context, entropy means the logarithm of the dimension of the Hilbert space of those states which can be given a field theoretic interpretation and thus localized inside the volume.

Thus, there is again a clear physical reason why the unsafe domain of moduli space cannot be mapped into the safe domain. Note again that matter obeying the holographic bound of [13] in the future, cannot alter the nature of the solutions near the true singularities.

To summarize: the U-duality group $\mathbb{G}_{10}$ divides the asymptotic domains of moduli space into three regions, corresponding to the spacelike and future and past timelike regimes of a Lorentzian manifold. Only the future lightcone can be understood in terms of weakly coupled SUGRA or string theory. The group theory provides an exact M-theoretic meaning for the Wheeler-De Witt Hamiltonian for moduli. Classical solutions of the low energy effective equations of motion with positive energy density for matter distributions lie in the timelike region of moduli space and interpolate between the past and future light cones. We find it remarkable that the purely group theoretical considerations of this section seem to capture so much of the physics of toroidal cosmologies.

4. Moduli spaces with less SUSY

We would like to generalize the above considerations to situations which preserve less SUSY. This enterprise immediately raises some questions, the first of which is what we mean by SUSY. Cosmologies with compact spatial sections have no global symmetries in the standard sense since there is no asymptotic region in which one can define the generators. We will define a cosmology with a certain amount of

SUSY by first looking for Euclidean ten manifolds and three form field configurations which are solutions of the equations of 11D SUGRA and have a certain number of Killing spinors. The first approximation to cosmology will be to study motion on a moduli space of such solutions. The motivation for this is that at least in the semiclassical approximation we are guaranteed to find arbitrarily slow motions of the moduli. In fact, in many cases, SUSY nonrenormalization theorems guarantee that the semiclassical approximation becomes valid for slow motions because the low energy effective Lagrangian of the moduli is to a large extent determined by SUSY. There are however a number of pitfalls inherent in our approach. We know that for some SUSY algebras, the moduli space of compactifications to four or six dimensions is not a manifold. New moduli can appear at singular points in moduli space and a new branch of the space, attached to the old one at the singular point, must be added. There may be cosmologies which traverse from one branch to the other in the course of their evolution. If that occurs, there will be a point at which the moduli space approximation breaks down. Furthermore, there are many examples of SUSY vacua of M-theory which have not yet been continuously connected on to the 11D limit, even through a series of "conifold" transitions such as those described above [22]. In particular, it has been suggested that there might be a completely isolated vacuum state of M-theory [23]. Thus it might not be possible to imagine that all cosmological solutions which preserve a given amount of SUSY are continuously connected to the 11D SUGRA regime.

Despite these potential problems, we think it is worthwhile to begin a study of compact, SUSY preserving, ten manifolds. In this paper we will only study examples where the three form field vanishes. The well known local condition for a Killing spinor, $D_\mu \epsilon = 0$, has as a condition for local integrability the vanishing curvature condition

$$R_{\mu\nu}^{ab} \gamma_{ab} \epsilon = 0 \,. \tag{4.1}$$

Thus, locally the curvature must lie in a subalgebra of the Lie algebra of $Spin(10)$ which annihilates a spinor. The global condition is that the holonomy around any closed path must lie in a subgroup which preserves a spinor. Since we are dealing with 11D SUGRA, we always have both the **16** and $\overline{\mathbf{16}}$ representations of $Spin(10)$ so SUSYs come in pairs.

For maximal SUSY the curvature must vanish identically and the space must be a torus. The next possibility is to preserve half the spinors and this is achieved by manifolds of the form $K3 \times T^7$ or orbifolds of them by freely acting discrete symmetries.

We now jump to the case of 4 SUSYs. To find examples, it is convenient to consider the decompositions $Spin(10) \supseteq Spin(k) \times Spin(10-k)$.

The **16** is then a tensor product of two lower dimensional spinors. For $k = 2$, the holonomy must be contained in $SU(4) \subseteq Spin(8)$ in order to preserve a spinor, and it then preserves two (four once the complex conjugate representation is taken into account). The corresponding manifolds are products of Calabi-Yau fourfolds with two tori, perhaps identified by the action of a freely acting discrete group. This moduli space is closely related to that of F-theory compactifications to four dimensions with minimal four dimensional SUSY. The three spatial dimensions are then compactified on a torus. For $k = 3$ the holonomy must be in $G_2 \subseteq Spin(7)$. The manifolds are, up to discrete identifications, products of Joyce manifolds and three tori. For $k = 4$ the holonomy is in $SU(2) \times SU(3)$. The manifolds are free orbifolds of products of Calabi-Yau threefolds and K3 manifolds. This moduli space is that of the heterotic string compactified on a three torus and Calabi-Yau three fold. The case $k = 5$ does not lead to any more examples with precisely 4 SUSYs.

It is possible that M-theory contains U-duality transformations which map us between these classes. For example, there are at least some examples of F-theory compactifications to four dimensional Minkowski space which are dual to heterotic compactifications on threefolds. After further compactification on three tori we expect to find a map between the $k = 2$ and $k = 4$ moduli spaces.

To begin the study of the cosmology of these moduli spaces we restrict the Einstein Lagrangian to metrics of the form

$$ds^2 = -dt^2 + g_{AB}(t)dx^A dx^B \,, \tag{4.2}$$

where the euclidean signature metric g_{AB} lies in one of the moduli spaces. Since all of these are spaces of solutions of the Einstein equations they are closed under constant rescaling of the metric. Since they are spaces of restricted holonomy, this is the only Weyl transformation which relates two metrics in a moduli space. Therefore the equations (2.7) and (2.8) remain valid, where G_{ij} is now the De Witt metric on the restricted moduli space of unit volume metrics.

It is clear that the metric on the full moduli space still has Lorentzian signature in the SUGRA approximation. In some of these cases of lower SUSY, we expect the metric to be corrected in the quantum theory. However, we do not expect these corrections to alter the signature of the metric. To see this note that each of the cases we have described has a two torus factor. If we decompactify the two torus, we expect a low energy field theoretic description as three dimensional gravity coupled to scalar fields and we can perform a Weyl transformation so that the coefficient of the Einstein action is constant. The scalar fields must have positive kinetic energy and the Einstein term must have its conventional sign if the theory is to be unitary. Thus, the decompactified moduli space has a positive metric. In further compactifying on the two torus, the only new moduli are those contained in gravity, and the metric on the full moduli space has Lorentzian signature.

Note that as in the case of maximal SUSY, the region of the moduli space with large ten volume and all other moduli held fixed, is in the future light cone of any finite point in the moduli space. Thus we suspect that much of the general structure that we uncovered in the toroidal moduli space, will survive in these less supersymmetric settings.

The most serious obstacle to this generalization appears in the case of 4 (or fewer) supercharges. In that case, general arguments do not forbid the appearance of a potential in the Lagrangian for the moduli. Furthermore, at generic points in the moduli space one would expect the energy density associated with that potential to be of order the fundamental scales in the theory. In such a situation, it is difficult to justify the Born-Oppenheimer separation between moduli and high energy degrees of freedom. Typical motions of the moduli on their potential have frequencies of the same order as those of the ultraviolet degrees of freedom.

We do not really have a good answer to this question. Perhaps the approximation only makes sense in regions where the potential is small. We know that this is true in extreme regions of moduli space in which SUSYs are approximately restored. However, it is notoriously difficult to stabilize the system asymptotically far into such a region. This difficulty is particularly vexing in the context of currently popular ideas [24]-[26] in which the fundamental scale of M-theory is taken to be orders of magnitude smaller than the Planck scale.

5. Discussion

We have demonstrated that the modular group of toroidally compactified M-theory prescribes a Lorentzian structure in the moduli space which precisely mirrors that found in the low energy effective Einstein action. We argued that a similar structure will be found for moduli spaces of lower SUSY, although the precise details could not be worked out because the moduli spaces and metrics on them generally receive quantum corrections. As a consequence the mathematical structure of the modular group is unknown. Nonetheless we were able to argue that it will be a group of isometries of a Lorentzian manifold. Thus, we argue that the generic mathematical structure discussed in minisuperspace[4] approximations to quantum cosmology based on the low energy field equations actually has an exact meaning in M-theory. We note however that the detailed structure of the equations will be different in M-theory, since the correct minisuperspace is a moduli space of static, SUSY preserving static solutions.

The Lorentzian structure prescribes a natural arrow of time, with a general cosmological solution interpolating between a singular past where the holographic principle is violated and a future described by 11D SUGRA or weakly coupled string theory where low energy effective field theory is a good approximation to the gross features of the universe. Note that it is *not* the naive arrow of time of any given low energy gravity theory, which points from small volume to large volume. Many

[4] A term which we have avoided up to this point because it is confusing in a supersymmetric theory.

of the safe regions of moduli space are singular from the point of view of any given low energy effective theory. We briefly described how those singularities are avoided in the presence of matter.

We believe that the connections we have uncovered are important and suggest that there are crucial things to be learned from cosmological M-theory even if we are only interested in microphysics. We realize that we have only made a small beginning in understanding the import of these observations.

Finally, we want to emphasize that our identification of moduli spaces of SUSY preserving static solutions of SUGRA (which perhaps deserve a more exact, M-theoretical characterization) as the appropriate arena for early universe cosmology, provides a new starting point for investigating old cosmological puzzles. We hope to report some progress in the investigation of these modular cosmologies in the near future.

Acknowledgments

We are grateful to Greg Moore and Ed Witten for valuable discussions. The work of Tom Banks and Luboš Motl was supported in part by the DOE under grant number DE-FG02-96ER40559. The work of Willy Fischler was supported in part by the Robert Welch Foundation and the NSF under grant number PHY-9219345.

A. The appearance of Γ_8 and $\Gamma_{9,1}$

In this appendix we will explain the appearance of the lattices Γ_8 and $\Gamma_{9,1}$ in our framework. For the group $\mathbb{G}_k$ we have found a bilinear invariant

$$\vec{u}\cdot\vec{v} = (9-k)\sum_{i=1}^{k}(u_i v_i) + \left(\sum_{i=1}^{k} u_i\right)\left(\sum_{i=1}^{k} v_i\right). \tag{A.1}$$

Now let us take a vector

$$\vec{v} \equiv \vec{v}_{123} = \left(-\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}, +\frac{1}{3}, +\frac{1}{3}, \dots\right) \tag{A.2}$$

and calculate its scalar product with a vector $\vec{p}$ according to (A.1). The result is

$$\vec{v}\cdot\vec{p} = (9-k)\sum_{i=1}^{k}(v_i p_i) + \frac{k-9}{3}\left(\sum_{i=1}^{k} p_i\right) = (k-9)(p_1+p_2+p_3). \tag{A.3}$$

Thus for $k=9$ the product vanishes and for $k=10$ and $k=8$ the product equals $\pm(p_1+p_2+p_3)$. If the entries $(-2/3)$ are at the positions i,j,k instead of $1,2,3$, we get $\pm(p_i+p_j+p_k)$. Substituting $\vec{v}_{ijk}$ also for $\vec{p}$, we obtain

$$\vec{v}_{ijk}\cdot\vec{v}_{ijk} = 2(9-k)\,. \tag{A.4}$$

So this squared norm equals ± 2 for $k = 8, 10$. More generally, we can calculate the scalar products of any two v_{ijk}'s and the result is

$$\frac{\vec{v}_{ijk} \cdot \vec{v}_{lmn}}{9-k} = \begin{cases} +\frac{2}{3} + \frac{2}{3} + \frac{2}{3} = +2 & \text{if } \{i,j,k\} \text{ and } \{l,m,n\} \text{ have 3 elements in common,} \\ +\frac{2}{3} + \frac{2}{3} - \frac{1}{3} = +1 & \text{if } \{i,j,k\} \text{ and } \{l,m,n\} \text{ have 2 elements in common,} \\ +\frac{2}{3} - \frac{1}{3} - \frac{1}{3} = 0 & \text{if } \{i,j,k\} \text{ and } \{l,m,n\} \text{ have 1 element in common,} \\ -\frac{1}{3} - \frac{1}{3} - \frac{1}{3} = -1 & \text{if } \{i,j,k\} \text{ and } \{l,m,n\} \text{ have no elements in common,} \end{cases} \tag{A.5}$$

so the corresponding angles are $0°$, $60°$, $90°$ and $120°$. Now a reflection with respect to the hyperplane perpendicular to $\vec{v} \equiv \vec{v}_{ijk}$ is given by

$$\vec{p} \mapsto \vec{p}' = \vec{p} - 2\vec{v}\, \frac{\vec{p} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \tag{A.6}$$

and we see that for $k \neq 9$ it precisely reproduces the 2/5 transformation (3.2).

Let us define the lattice Λ_k to be the lattice of all integer combinations of the vectors $\vec{v}_{ijk}$. We will concentrate on the cases $k = 8$ and $k = 10$ since Λ_k will be shown to be even self-dual lattices. Thus they are isometric to the unique even self-dual lattices in these dimensions.

It is easy to see that for $k = 8, 10$, the lattice Λ_k contains exactly those vectors whose entries are multiples of 1/3 equal modulo 1. The reason is that all possible multiples of 1/3 modulo 1 i.e. $-1/3, 0, +1/3$ are realized by $-\vec{v}_{ijk}, 0, \vec{v}_{ijk}$ and we can also change any coordinate by one (or any integer) because for instance

$$\begin{aligned} \vec{v}_{123} + \vec{v}_{456} + \vec{v}_{789} &= (0,0,0,0,0,0,0,0,0,1)\,, && \text{for } k = 10\,, \\ \vec{v}_{812} + \vec{v}_{345} + \vec{v}_{678} &= (0,0,0,0,0,0,0,-1)\,, && \text{for } k = 8\,. \end{aligned} \tag{A.7}$$

Since the scalar products of any two $\vec{v}_{ijk}$ are integers and the squared norms of $\vec{v}$'s are even, Λ_8 and Λ_{10} are even lattices. Finally we prove that they are self-dual.

The dual lattice is defined as the lattice of all vectors whose scalar products with any elements of the original lattice (or with any of its generators $\vec{v}_{ijk}$) are integers. Because of (A.3) it means that the sum of any three coordinates should be an integer. But it is easy to see that this condition is identical to the condition that the entries are multiples of 1/3 equal modulo 1. The reason is that the difference $(p_1+p_2+p_3)-(p_1+p_2+p_4) = p_3-p_4$ must be also an integer – so all the coordinates are equal modulo one. But if they are equal modulo 1, they must be equal to a multiple of 1/3 because the sum of three such numbers must be integer.

We think that this construction of Γ_8 and $\Gamma_{9,1}$ is more natural in the context of U-dualities than the construction using the root lattice of E_8 with the $SO(16)$ sublattice generated by $\pm e_i \pm e_j$.

Let us finally mention that the orders of $\mathbb{G}_3$, $\mathbb{G}_4$,... $\mathbb{G}_8$ are equal to $2 \cdot 3!$, $5 \cdot 4!$, $16 \cdot 5!$, $72 \cdot 6!$, $576 \cdot 7!$ and $17280 \cdot 8!$. For instance the group $\mathbb{G}_4$ is isomorphic to S_5.

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We can see it explicitly. Defining

$$p_i = R_i - \frac{1}{3}(R_1 + R_2 + R_3 + R_4 - R_5)\,, \qquad i = 1, 2, 3, 4\,, \tag{A.8}$$

which leaves p_i invariant under the transformation $R_i \mapsto R_i + \lambda$, the permutation of R_4 and R_5 is easily seen to generate the 2/5 transformation on p_1, p_2, p_3. Note that $p_1 + p_2 + p_3 = -R_4 + R_5$.

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